Technology and the Laws of Thought
Part 2

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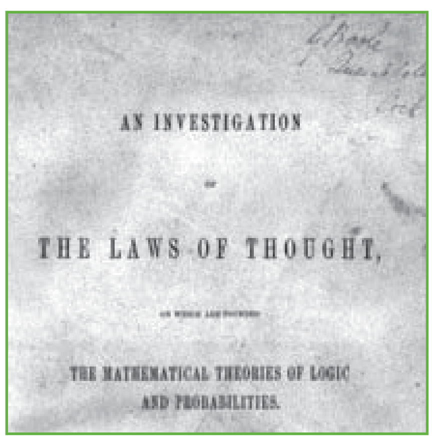
The Laws of Thought

> *No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it until it also gives the impression of being beautiful.*
>
> – George Boole

The dawn of the 19th century saw a rapid development of natural science, leading to the rise of new ideas in physics and mathematics. Industries were in full swing, laws of electricity and magnetism began to be investigated (by the likes of Faraday, Volta, and Ampere), and non-Euclidean geometry was developed by many mathematicians, including the "Prince of Mathematics," Carl Gauss. Devotion to architecture of buildings was now transferred either to technology or to a study of the architecture of the human body, leading to large strides in anatomy and physiology. Cell theory took shape, and the possibility of different functions of the body being localized in different parts of the brain began to be investigated.


AN INVESTIGATION

THE LAWS OF THOUGHT,

THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES.

It was precisely in the midst of this environment that George Boole (1815–1864) lived, and his life shows the indications of the struggle between the different aspects of the thinking process. Deeply religious by nature, Boole had a mystical experience in 1833, later described by his wife, Mary Everest Boole:

> My husband told me that when he was a lad of seventeen a thought struck him suddenly, which became the foundation of all his future discoveries. It was a flash of psychological insight into the conditions under which a mind most readily accumulates knowledge [...] For a few years he supposed himself to be convinced of the truth of the Bible as a whole, and even intended to take orders as a clergyman of the English Church. But by the help of a learned Jew in Lincoln, he found out the true nature of the discovery which had dawned on him. This was that man's mind works by means of some mechanism which functions normally towards Monism. (M.E. Boole, *Indian Thought and Western Science in the Nineteenth Century*, 1931)

The inspiration for inner effort is once more seen to originate in the sphere of religion, as borne out by Boole's deep interest in all religions and his desire to work in the cause of "pure religion." Studying his life and conceptual development must hence necessarily include the logical precision, the aesthetic sense, as well as religious devotion simultaneously, as it existed in the individual. This is not usually done today, and there is a recurring tendency to pick and choose. Only one part of the story is valued and the rest discarded as being irrelevant (quite against the spirit of scientific investigation):

> The father of pure mathematics, as Bertrand Russell would later refer to Boole, had not

> been purely interested in mathematics, nor was his mathematics free of the "impurities" of extradisciplinary concerns, in particular, religious ones. The symbolic logic that is now the essential tool for secular philosophers and that forms the basis for dispassionate computers began in the mind of a warm-blooded, religiously concerned idealist. (Dan Cohen, *Equations from God*, Ch. 3, 2007)

In other words, the assumption is made that mathematics is "disciplinary" and everything else is necessarily a separate box or an "impurity." This assumption is one of the primary reasons why this aspect of Boole's life and work is hardly known or even realized today, making it a fact worthy of mention.

As already described in Part 1 of this series (*Research Bulletin* 21-2, Autumn/Winter 2016), Bacon pioneered the use of binary to represent letters and language. Leibniz developed the use of binary numbers in mathematics and suggested their use in calculating machines; he also suggested that logic might be represented mathematically. Boole took the first steps to accomplish that by trying to represent thought and logic itself in a binary form. Language, Mathematics, and Logic—these were the domains that could now be represented in a binary form, as a work of the Philosophers. It is clear that there are a finite number of letters in the alphabet, and a finite representation of numbers. Thus, letters and numbers can be represented in binary, a feat which does not alter the very content of thought itself. For instance, I can express a number in any base, and I can also write a certain word in any script. The form of the letter is not crucial to understanding the meaning of the communication. However, with Boole, the notion of "meaning" and logical thinking itself is expressed in the form of binary algebra, which has to be analyzed further.

Boole considered his work a natural extension of the works of Aristotle. Prior to Boole's analyses, logic was developed as an interrelationship of concepts or propositions, which were then differentiated. Qualifiers such as "all objects," "some objects," and "no objects" were referred to as *subjects*, while their qualities, e.g., "round," "white," etc., are called *predicates*. The verb forms the link or *copula*. The copula was seen as a logical connecting link, and it is normally a verb. In other words, the bridge between the subject and predicate involved either an existence (*is*) or lack of existence (*is not*). Together, different kinds of propositions were created, such as:

> All diamonds are solid.
> Some stones are soft.
> Some pebbles are not black.
> No rocks are liquid.

Logical propositions were studied, discussed, debated and elaborated by the Greeks of the same period, the Arabs of the 7th–10th centuries, and the Scholastics of the 13th century. Thus, the works of Aristotle on logic have had an immense role in the development of further thought for nearly two millennia. Nevertheless, the development of a logical train of thought itself was not altered, and everything was still expressible in terms of basic syllogisms such as this one:

> All diamonds are solid.
> Kohinoor is a diamond.
> ∴ Kohinoor is solid.

This is essentially the crux of the development of logic, and is also called a *syllogism* or *deduction*. A logical deduction was therefore at the root of all philosophy for many centuries, as workers in this field strove to build thought itself and also the relationship between man and the world. Rules were derived for these logical deductions and for determining validity and invalidity of propositions. The laws of thought according to this system are:

A is A
Law of Identity: A concept is equal to itself.

A is not (not A)
Law of non-contradiction: A concept is not the same as its opposite.

All A is either A or is not A
Law of excluded middle: A concept is either true or false.

These were followed quite strictly by medieval logicians. For example, Avicenna (10th century) is said to have declared that: "Anyone who denies the law of non-contradiction should be beaten and burned until he admits that to be beaten is not the same as not to be beaten, and to be burned is not the same as not to be burned." A dry sense of humor indeed, but it is hard to imagine today how central these debates were to the intellectual life of the time, when the ideas one held to be true had life-or-death repercussions.

It can be seen in the form of the copula that the laws of logic denote *static* situations, i.e., either something *is* or *is not*. There is no other possibility. It is interesting to probe the origins of these statements in logic and ask why the Greeks posed a statement in that particular fashion. Because the experience of one era determines the concepts woven out in the following period, the natural question to ask is: What motivated the laws of thought? What was the experience on which these laws were based?

A study of the thought life and pursuit of Truth of the ancient Greeks shows that geometry was revered, as was music (*Harmony of the Spheres*). It was understood that in geometry man could really grasp the structure of the world and man's place within it. This is indicated by the Platonic saying "God geometrizes" which was said to be written at the entrance of his School of Athens. In fact, Aristotle was a student of this school himself, and geometry formed the soil for the seeds of logic to be sown. For example, a shape is either a triangle or not a triangle: The same spatial experience could not take two separate forms at the same time. A triangle as a concept remained something static and did not change with time, giving an assurance of a firm foundation for thinking. This clarity of thought in geometry guided the early development of logic as well, resulting in a twofold copula: *is* or *is not*. The origin of the copula can be indicated as:

School of Athens =>
Geometry =>
Copula: *is*/*is not*.

Similarly, it can be observed that there are *quantifiers* in the statements: *all, some,* and *none*. Here, it is a matter of encompassing the element through number, which is necessarily what our notions of quantity are tied to. Whether or not an actual count is performed, *all* is necessarily more encompassing than *some*, and *some* is more encompassing than *none*. The origin of these relationships can be traced to the domain of arithmetic, or the positive real line. This was the development that had its roots in musical ratios and numbers. It is important to keep in mind that it is the form of the quantifier itself that is of interest here, and not the concept that it is referring to. For example, consider:

All ideas are wonderful.

It is not important whether or not the *ideas* are quantities, but that the notion of *all* itself is derived from the notion of a quantity. Quantities owe their origin to arithmetic. The origin of the laws of arithmetic can be traced back to a much earlier School: the Pythagorean. The Pythagoreans revered numbers above all as the guiding principles of the Universe, and their ideas had taken strong root by the time of Plato and the School of Athens. Geometry was also seen as being derived *from* arithmetic, paralleling their natural developments in the Schools of Pythagoras and Plato. Hence, the derivation of logical terms is:

School of Pythagoras =>
Arithmetic =>
Quantifier: *all/some/none*

School of Athens =>
Geometry =>
Copula: *is/is not*

It is well known that the Greeks were quite skilled in the development of both, and it comes as no surprise that logic owes its origins and internal structure to ideas from geometry and arithmetic.

There is yet another aspect of this logic that has to be addressed. This is the *subject and predicate*, which can be called *class* in general. Conventional understanding simply takes *class* as "collection of objects" or a sack of goods, plainly speaking. However, meanings for the same words can be quite different in different periods (an early example is perspective drawing). Hence, this does not take into account that in the Greek era, even ideas were treated as real "objects." Indeed, Plato's philosophical work involved the notion that true reality consists of ideas. In stark opposition to Plato, the Stoics of a later era considered the entire world including ideas as being corporeal, or physical. This also mirrored the different motivations of Platonists and Stoics: Platonists were concerned with concepts from the ideal world (metaphysical realities), while the Stoics focused on actual application within a deterministic physical world.

These polar opposites must be taken into account, as they are central to the operations of logic that developed later. If the name of a class of object or ideas is called "concept," the summary of Aristotelian logical forms can be written as:

Quantifiers: Arithmetic
Copulas: Geometry
Class: Concepts (objects/ideas)

The Stoics were more interested in the utility of the laws of thought to living a moral life than in deriving implications of the laws themselves. This meant that they focused on the class alone, and developed consequences. Naturally, Aristotelian logic involved these derivations of consequences, but they were not studied exclusively as done by the Stoics. It is therefore from the Stoics that sentences of this form are derived:

If A then B
Not both A and B

This is perfectly suited not only to develop consequences, but to determine actions. A and B can thus denote *actions*, in addition to ideas or objects. It also rests on a condition "if," which separates the true from the false, the "is" from the "is not." Since A and B can denote actions, the following statement is also possible.

If A then *do* B.

This then adds to the list of copulas, in a different form—not only can B *exist* or *not exist*, be true or false, but it can also be *done*. It is particularly well suited to applications. The updated summary becomes:

Quantifiers: Arithmetic
Copulas: Geometry (is/is not), *Actions (do)*
Class: Concepts (physical/metaphysical)

This brings the discussion from the realm of the Philosopher back to the realm of the Craftsman, and it is worthwhile to see that the logic of the Greeks had its renaissance only in the 19th century: the age of the Engineer/Scientist. In this era, Boole developed a logic that expressed only one particular form of the four Aristotelian copulas, viz., *equality*, represented by *equations*. The four statements of the Greeks are reduced to one expression relating to concepts.

All diamonds are solid. 1
Some stones are soft. 2

Some pebbles are not black. 3
No rocks are liquid. 4
=> Coal = Black

Since equations with variables are nothing but the formulation of *algebra*, for the first time logic is expressed through algebra, a branch of mathematics. Logic gets mathematized for the first time. This conversion of logical copulas to equality has been addressed in recent works:

> Where Aristotle saw predications, Boole saw equations. Boole realized that his theory of logical form was in radical opposition to Aristotle's, but he seems to have thought that Aristotle had just not gone deep enough, not that Aristotle was fundamentally mistaken. Boole's pattern was S–is–P, Subject–is–Predicate, or S=P, Subject equals Predicate. (Corcoran, THPL 24 p.261, 2003)

This conversion of four possible expressions into the single expression of equality had its consequences: What was immediate for Aristotle required mediation for Boole. Again, Aristotelian simplicity becomes Boolean complexity. For example, Aristotle would go from the two premises "Every square is a rectangle" and "Every rectangle is a polygon" immediately—in one step—to the conclusion "Every square is a polygon." Boole broke this down into eight tediously meticulous equational steps. The first step is going from the second premise, "Every rectangle is a polygon" to an intermediate conclusion arrived at by something analogous to multiplying equals by equals, namely "Every square that is a rectangle is a square that is a polygon," "multiplying" both sides of the equation by "square." (Corcoran, ibid.)

In other words, conversion of all the copulas into a single one (*is* or *equals*) made the representation of statements cumbersome. If only *equality* is included as a copula, there is naturally no inequality that can be expressed, leading to the complications just quoted. If Boole wanted to *extend* logic, a natural way forward would have been to study different copulas, which do not make sense in Aristotelian formulation. For example, compare these two sets of statements:

All men are mortal.	All children like candy.
Socrates is a man.	The cat likes children.
=> Socrates is mortal.	=>The cat likes candy.

The second syllogism on the right is not necessarily valid in real life, and it indicates that a different domain opens up when a different copula is used. Hence, a new form of logic can be determined for *likes* and *does not like* just as it was built up for *is* and *is not*. Instead, the set of copulas is reduced to just *is* by Boole.

This alteration had other effects as well. With the nature of the copula altered, the subjects, predicates and quantifiers changed. Boole reduced the quantifiers *all, some*, and *none* to just two: *all* (1) and *none* (0). Also the quantity, which stood by itself in the traditional system, was now inserted *with the class*. This means that one no longer had *all diamonds* but instead just a single *all-diamonds*, which became the same as just diamonds because *all* = 1. Similarly, instead of saying "*no* diamonds" one had to say "*no* all-diamonds." The quantifier gets attached to the class itself. Hence, just as the ancients had to create a number to represent zero, for the first time a new *class* or object also had to be created for the purpose of saying "nothing." Nothing becomes a class of objects! For example, the standard logical statement transforms like this:

No A is B.
∴ (*No* A) is (B).
∴ (All A) and (All B) is (nothing).
∴ A.B = 0

This is a phenomenal transformation, as the variable "zero" or "nothing" has literally been created *ex nihilo* to fill in the function of *is not*. Because the copula is no longer available, the object itself must have "nothing."

These laws developed by Boole were complemented and enlarged by Gottlob Frege (1848–1925). Like the Stoics before him, Frege concentrated on creating a system where all logical statements can be expressed symbolically, and also applied. Hence, he elaborated propositional logic in a mathematical form, making it possible not only to express equations mathematically, but also to include the application using propositional statements (*if… then, and, or,* etc.). This enabled logic to be expressed as a *mechanism,* an *action*. What Boole did for Aristotle, Frege accomplished for the Stoics, and the works of these two thinkers and their contemporaries (such as C.S. Peirce) form the basis of what is known as Boolean algebra today.

If all the quantifiers of traditional logic had been included, it would never have been possible to indicate it by a machine, because of the quantifier *ALL.* As hinted earlier, these quantifiers bear a direct resemblance to arithmetic, represented like this:

NONE	0
SOME	1
ALL	∞

No mechanism can generate infinity, so that had to be removed. The meaning of *ALL* is shifted onto "1," giving:

NONE	0
ALL	1

The quantifier *SOME* is no longer required: It is simply absorbed in the class name itself. Therefore, both the names of classes (subject/predicate) as well as the copula (*is not*) become *quantified*. This conversion of all logical statements into algebraic form is thus seen to remove everything that could not be quantified or mechanized and retain only that which could. It is not an extension, as Boole believed, but a reduction, a filtration.

Reductions of logical statements into mathematical expressions make them well suited to include both calculations and the rules of calculations into the mechanism of a device. Sure enough, this generated the possibility of replacing the will element of thinking and to outsource it to the machines:

> With earlier sorts of logic, to determine whether the validity of an argument had been proved, we still needed to understand the meanings of words—at least *some* of the words—and this meant we still need to *think*. With symbolic logic, by contrast, we don't need to think at all, or rather the only thing we need to think about is whether the symbols appear in the order specified by the rules that govern them. In consequence, so long as the proof is sufficiently spelled out in a symbolic language, the task of verifying it is strictly clerical. What we look at is simply a matter of form—and *purely* form. … In this respect, then, determining whether an argument's validity has been proved within the system is strictly mechanical, and, in our age, it can certainly be done by machines. (Shenefelt and White, *If A then B, How Logic Shaped the World*, p.252, 2013)

Thus, "formal" logic served to reduce the necessity for thought and to increase the mechanization of concepts.

At this point, it is worth asking: Was this "mathematization" necessary? In other words, what prior reason could one have to insist that logic has to be mathematical and, more specifically, algebraic? What is the reasoning that motivated this logical development? A study of Boole's works gives a surprising answer to this:

> Whence it is that the ultimate laws of Logic are mathematical in their form; why they are, except in a single point, identical with the general laws of Number; and why in that particular point they differ;—are questions

> upon which it might not be very remote from presumption to endeavor to pronounce a positive judgment. Probably they lie beyond the reach of our limited faculties. (Boole, *Laws of Thought*, Ch.1)

Thus, there is no specific reason for insisting that logical rules have to correspond to mathematics, other than the general feeling of confidence that thinkers have in the methods of mathematics (perhaps due to the great success of the Industrial Revolution) as exemplified by Boole. It is a *claim* and not a deduction from previous circumstances nor a solution to a specific problem. This fact is important to highlight as it shows that the factors motivating the *creation* of mathematical logic were *neither mathematics nor logic*.

Boolean algebra connected logic to algebra, in the process significantly altering the form of logic as it was practiced for centuries. Several features of logic were extended, but only in terms of their algebraic representations (logical gates such as AND, OR, NOT, etc.) At the same time, quantifiers are reduced from three to two, copulas are reduced from two to one, and new categories of "everything" (1) and "nothing" (0) are introduced as classes. Language and verbal influences are removed entirely and replaced by abstract symbols. Thus, while the theory itself was generated due to the *religious* will-temperament of its creator, the effect of it was to create *mechanical* (will) equivalents of logical statements. *Religious* will inspired *mechanical* will.

Traditional logic was guided by geometry and arithmetic, which were in turn guided by experience. Replacing this by an "algebra" of logic removes those restrictions of experience, giving it a free reign in terms of equations and combinations. Hence, Boolean algebra *appears* to show an extension of logic, even though it is a reduction of it to include only a subset of mathematical concepts.

Mathematics is logical. But now, since logic and algebra were combined as symbolic logic, this form of logic had an effect in turn on mathematics itself. Hence, mathematics itself is changed and seen differently. Because mathematics is at the core of science, this impacted all of natural science.

Numbers and Neurons

> *Music is the pleasure the human mind experiences from counting without being aware that it is counting.*
> – Gottfried Leibniz

So far, the transformations occurring in logic and their relationship to mathematics have been described. In particular, the key changes that occurred with Boolean algebra have been indicated, viz., the transformation of the quantifier "all" from infinity to one, the removal of the copula "is not," and the addition of the concept "nothing" into the predicate or class name. Frege's work helped to make this logic practical by retaining the copula "do" and expressing a logical statement as a sequence of abstract operations. That which earlier required a knowledge of language, and hence an evaluation of meaning, was converted into an automatic operation. In addition, the sequence of development of logic was now reversed:

Mathematics and Arithmetic =>
Geometrical understanding =>
Logic of Ancient Greek (ca 200 BC)

Logic of Ancient Greek =>
Algebraic expressions =>
Mechanical operations (ca 1900 AD)

In other words, instead of deriving logic *from* mathematical understanding, it was *equated* to a specific mathematical domain. This logic, called *symbolic logic*, now became identical with mathematics. It is but natural to go one step further, and that is precisely what happened: A new mathematics was derived from symbolic logic. Development of mathematics hence came full circle, from mathematics generating logic to symbolic logic generating formal mathematics.

Mathematics =>
Greek Logic =>
Algebraic Logic =>
Formal Mathematics (200 BC to 1900 AD)

As indicated earlier, while engaging the thinking process, there is a direct perception of a force, or will element, in the effort needed to form a thought. Without this will element, there is no difference in thinking about something for the first time and thinking about something that has already been thought about. It has also been indicated that only the repetitive aspect of this force can be directed to a machine. An example can be used to study this process in some more detail.

Assume that you are asked to derive the surface area of a cone for the first time. The first struggle will probably be the hardest, as you wend your way using calculus to determine the right formula. Now, if the same question is given again, and you are asked to demonstrate to a third party, the derivation flows more smoothly, and the effort involved is significantly less. After a few repetitions, there is minimal effort involved, and the process becomes automatic and, as a consequence, easier to repeat. Compare this to the effort involved in lifting, say, a heavy rock. The first time, there is a large effort involved. After several repetitions, involving perhaps weeks, months or even *years* of experience, it gets easier to lift. In the process, both the bones and the muscles are strengthened. Thus, mental effort appears to be a greatly accelerated analog of physical effort. Both have the quality that, with repetition, inner strength increases so as eventually to match the resistance. Once the strengths are matched, the task becomes easier and, in the end, becomes mechanical. Thus, mechanical process is seen to be the very minimal residual form of inner effort.

Secondly, after receiving the problem of a cone, assume that you are given a problem in properties of prime numbers. Naturally, this would refresh the will-element once again. After solving this second problem, suppose you declare that it took the same amount of effort to solve this problem as it took to solve the one on surface area. Nevertheless, doing the problem of surface area again would not involve the same effort, but doing a *new* problem requires roughly the same effort. This clarifies the process some more: *Novelty* is the key criterion that demands effort each time, while *identical repetition* reduces effort over time. Therefore, in terms of the thinking process as a whole, including these willing and feeling aspects, the capacity to understand involves the capacity to generate new ideas, as opposed to repeating existing ideas. This freshness of thinking is developed when trying to understand topics that are as different from each other as possible or when generating new ideas for the same topic.

Hence, a healthy thought process necessarily involves creating new ideas. But what is the result of ideas being restricted to those that are logical, algebraic, and, hence, mechanical? The natural consequence of that restriction is that ideas themselves are seen as being mechanisms! In a direct inheritance of Stoic philosophy, whose *assumption* was that the whole world is corporeal, thoughts are seen as mechanisms. Here it is easy to observe how the cause and effect are inverted. The idea of mechanism is one arrived at through many centuries of effort of both Craftsmen and Philosophers, and now it is claimed that thought is a mechanism. It is easy to see how one gets into a knot when pursuing the topic like this, when one asks, "What is a mechanism?" Answering this would require thinking once more, looping back to square one. This is reminiscent of the old joke:

> *What is mind?* Does it matter?
> *What is matter?* Oh, never mind!

An illustration of the fact that mechanism is not the same as understanding a concept is shown by the "Chinese Room Argument," described by John Searle in 1980.

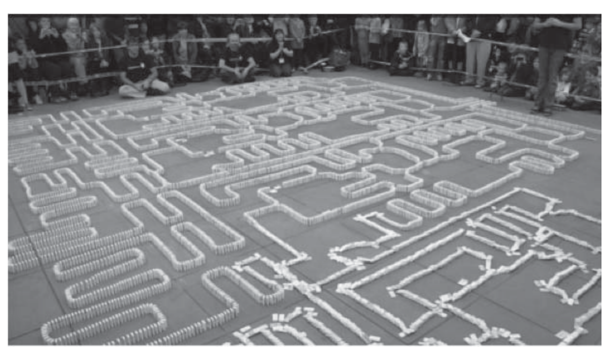
If you see this shape, "什麼" followed by this shape, "帶來" followed by this shape, "快樂"	then produce this shape, "爲天" followed by this shape, "下式".

Translator in the room, with his translator (Illustration courtesy: David L. Anderson, CCSI)

In this argument, a person who has does not know the Chinese language is kept in an enclosed room and asked to read off a certain set of characters on the input screen. He is given a big, fat rule book, containing all the rules about what the output is for any specific input character set. It is possible for this Translator to generate a response for any string of shapes, and, except for the fact that it would take him a while to provide the response, it would be indistinguishable from someone who knows Chinese. Yet, in spite of the fact that he performs accurate pattern recognition, the Translator knows no Chinese and does not understand what any word means at the end of the day. Meaning is completely missing, as the pattern is *not* the *meaning*. This method of functioning is also seen historically with the first major computing machines: looms that wove patterns of cloth.

This is a direct result of the application of Frege's rules for generating symbolic logic, which can be executed mechanically. As a matter of fact, instead of keeping an actual person inside the box, it is possible to replace it with an elaborate pattern of dominoes. All that would be required for repeated operation is that a fallen domino (or sets of fallen dominoes) can be made to stand again by pressing a lever outside the room, or by a falling domino inside the room. In this case, the "rules" are formed by the pattern in which the dominoes fall. By connecting each Chinese character to a specific set of dominoes, which either fall or do not fall, it is possible to generate a response at the output that once more corresponds to a specific character. As anyone who has arranged dominoes for entertainment knows, kicking a domino can hardly be called "thinking."

Whether it is with dominoes or with gears and levers, the nature of mechanism is clearly seen as something that is necessarily



(Courtesy: Matt Parker's Domino demo)

deterministic and finite. The placement of the thinking process is also not clearly identified in the case of a mechanical device. Normally it is taken for granted that the machine helps to think once it is set into operation. This idea was suggested earlier, that the repetitive element of thinking might be outsourced to the machine. In reality, the situation is exactly reverse: Thinking is involved only in setting up the "rules" of the operation or in preparing the rule book or the domino arrangement—and it stops the moment the machine is set in motion. Whether one sees dominos falling, gears churning, or electricity firing, the process is identical in essence. It is just like kicking a stone downhill and watching its progress according to gravity. Hence, the basic process underlying computation is mechanism and not thought.

Another consequence of this restriction of logical form to mechanism, especially with the copula and quantifiers, is the effect on the notion of infinity. As already pointed out, geometry motivated the copula, and arithmetic motivated the quantifiers in Greek logic. However, several changes occurred in these subjects toward the middle of the 19th century, simultaneous with the works of Boole. Just a year prior to Boole's *Laws of Thought*, William Hamilton discovered the algebra of quaternions (with one real and three complex quantities).

This topic was added to the already existent serious debates among mathematicians as to the nature of numbers, particularly negative and complex numbers, which could not be grasped intuitively. Hence, as algebra was undergoing a massive change, geometry was also being revamped from the ground up: The Euclidean system that had stood for two millennia, just like logic, was called into question. The postulate that states that parallel lines never meet was found to be unnecessary for a consistent geometry, throwing open the door to Non-Euclidean Geometry and Synthetic Geometry (also called Projective Geometry) that began to be developed as generalizations of Euclidean Geometry. In this form of Geometry, parallel lines do meet—at a point at infinity.

Frege was strictly against the replacing of the axioms of Euclid and, instead, added *more* axioms to make the system stricter. He declared that "no man can serve two masters," i.e., either Euclidean or non-Euclidean systems were true (*On Euclidean Geometry*, p.251, 1997.) Since the assumptions taken to be "self-evident truths" were no longer valid, the reaction was to abandon the idea and focus on setting up a set of abstract axioms, whose consequences were explored.

Around the same time of Frege's work (1879), irrational numbers were clearly defined by Richard Dedekind (1872), while Georg Cantor explored the fact that there are an infinite number of possible infinities (1874). However, the mathematization of logic was restricted to the Euclidean domain, giving rise to analog machines. Moreover, only whole numbers (zero and positive integers) could be represented in binary code, which led the way to digital machines. These were two immediate consequences of the rejection of both projective geometry and infinity–a fact that will be described more in the next chapter.

While infinity is not accessible to mathematical representation, one can still express it as a concept, as it is done in this very sentence. Hence, the logical domain extends beyond that of finite numbers. It is necessary to see that logic is also not restricted by the use of copulas *is* and *is not*. This opens the door to include non-mathematical ideas (or at least, the non-Euclidean) into logical form and to alter the form of analyzing the possibilities in a situation. Hence, thinking has a range that intrinsically exceeds that of mathematics and mechanics.

This fact is mostly overlooked by most comparisons of the mind and the machine. Consider one of the typical examples used for estimating the power of mind and machine: a game of chess. This game has highlighted the epitome of intellectual capacity for several

McGonagall's giant chess set

centuries. It is possibly one of the few games today that is simple and yet represents so well strategy, combinatorics, even elegance and intuition.

Traditionally, in order to investigate whether or not the machine can compete with the human mind, the number of possible moves is calculated and probabilities weighed in order to determine outcomes. Orderly to begin with, a rough estimate of possible games that can ever be played on the chessboard rises astronomically with every move: 400 board configurations after the first move, nearly 200,000 after the second move, up to 121 million after the third, and estimated to be nearly 10^{100000} in total (no precise calculations exist). This is just with 64 squares and 32 pieces. Because this appears to be a very large, yet finite number, it is likely that inevitably, with the rise in the sheer computing power of machines, a machine might beat a human player. This is the method followed when comparing the mind and the machine: The possible number of combinations is calculated in each case.

However, a second look at the game reveals an inherent assumption in this calculation: *that the rules of the game remain fixed*. Because all the rules are geometrical/arithmetic rules (directions and steps of movement) that can be quantified with just an array of integers, the logic used is precisely that which can be mechanized. What if, instead of restricting logic the way it is usually done, the numbers of possible rule-configurations are calculated? If rules are allowed to change, how would it alter the situation? In this case, one can easily see that the calculations are quite impossible to do, simply because there is *infinity* of rules to choose from. Just as young children make up rules and change them as they play along, rules can always be altered. This option opens the door to the full range of human mental capacities to act and removes the barrier that existed for the "rules of the game." It does not matter if there are 64 squares and 32 pieces or 64 squares and 31 pieces—an infinite number of games is possible if the rules can be changed. As Georg Cantor rightly pointed out, there are an infinite number of infinite numbers. Regardless of the number of squares or pieces, "the sky is the limit."

With just 26 letters of the alphabet, the English language has been growing and continues to grow with every added speaker. Countless numbers of books have been written, poems composed, reports described, with no end in sight. With a few notes of the musical scale, music is still being composed afresh every day. With a few colors, new paintings are created every day. Creativity, therefore, is not limited by the tools of expression or number, but only by the capacities of the thinking and intuitive processes. In numerical terms, a sustained creativity occurs *ex nihilo, infinitum*.

This capacity of the rules to change completely alters the way in which calculations are done and ends up making the calculations irrelevant. As long as the rules for calculations are constant, they can carry some meaning, but what meaning can be obtained when the rules themselves get changed? This explains the difficulty faced by the best of mathematicians when dealing with things like negative numbers and imaginary numbers, because the rules of mathematics are changed.

This possibility of changing rules was missed almost completely by logicians of the 19th century. It is possible that many aspects can now be seen much better in hindsight due to greater familiarity with the nature of technology, while logicians and mathematicians of the 19th century were still experiencing the beginning of the Industrial Age. Biology had already reached a high level of development at the end of the 19th century, with the detailed study of cell structures and the nervous system being conducted. With the advent of technology, cells were seen as tiny machines or tiny test tubes with chemical interchanges with the DNA. The brain itself, made up of neurons, was seen in a similar fashion to resemble a gigantic factory-like operations-control center. Biology was explored using the tools of physics and chemistry, where mechanism reigned supreme, and it began to be taken for granted that cell functions determined human behavior.

Removing the restrictions of number gives a much clearer look at the activity of the brain. Just as the piano can be broken or a string of a violin can go missing, it is indeed possible that physical variations in the brain cause intellectual capacities to vary. Snapping a few keys or over-tightening a few strings affects the sort of sounds the piano can produce. Thus, there is definitely a connection to the physical instrument and its expression. But the possible tunes that can be created on the broken piano are once more infinite; hence there can be no calculable difference between the working piano and the broken one, in terms of its music. In addition, the piano works due to the laws of mechanics: vibration, lever action, pushes and pulls, pressing of keys, etc. Everything can be calculated, as to how each key can be pressed and each sound produced. *It is the number of possible tunes that is not calculable*. Modern biology focuses entirely on electrical actions and cell behavior, just as if a person who wanted to learn music were to take the piano apart piece by piece to determine how the sound is created. This would naturally lead nowhere as far as music is concerned, since music lies in the combinations of sounds, not in their mechanisms. Similarly, creative thought lies in its capacity to generate fresh ideas, and not in amino acids, neurotransmitters, and network interconnections. This shows the key difference that takes place in calculating the capacity of human thought and that of machines.

It might be argued that the piano has a player who affects it from outside, and there is none who can be observed to act as the "player" for the brain. Here is where internal observation comes into play, the very same observation that enables one to observe one's own thought processes and define notions such as "consistency," "logical," and "internal effort." Ideas of logic and concepts do not occur by peering through a microscope, but directly from an individual's experience of them, which are later confirmed by looking through any instrument. Therefore, there is indeed a player, *the human individual*, whose thoughts leave an impression on the brain via electrical impulses in the same way a piano player leaves his impressions on the piano via the keys.

If numbers *have* to be used to get an idea of the difference between musical instruments and the human brain, consider this: Music consists of a few basic notes, and there are new tunes being produced even today. Calculating the number of songs ever sung would literally take forever and is infinite. If a few octaves of notes have been the basis of all the infinite music ever played and will be played on instruments, what indeed could be the capacity of 100 *billion* neurons? That gives the real qualitative comparison of human creative capacity, as a quantitative one is not possible.

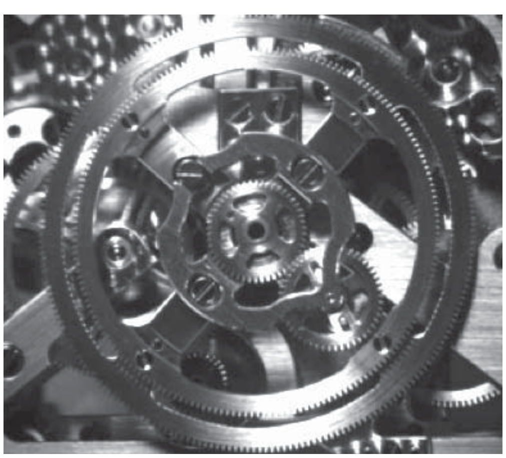
The Digital Transition

> *Before Turing, things were done to numbers. After Turing, numbers began doing things.*
> – George Dyson, *Turing's Cathedral*

Mathematics and logic form the mold into which technological applications are cast. As brought out in the previous chapter, far-reaching modifications were made to logic, and this effect carried through into mathematics as well. The way in which this change in logic, and the corresponding change in mathematics, determined the range and capacity of further technological development will be the focus of this chapter.

As the 20th century dawned, mathematicians in Europe continued work on formal mathematics, attempting to define a fully consistent set of axioms based on the rules of symbolic logic. This transition from self-evident truths to axioms as a basis for mathematics was completed with the monumental work by Bertrand Russell and A.N. Whitehead: *Principia Mathematica* (1910). At the precise point that mathematics transitioned into the abstract realm, technology came to the fore in a very real and powerful way, via World War I.

The Great War, as it was called, was the first one in recorded history that depended heavily on technology. Trench warfare led nowhere, shifting the emphasis onto bigger and better guns. The use of mechanical calculators received a huge boost due to the need for accurate firing ranges of missiles and cannons. Even though digital machines such as Herman Hollerith's mechanical census tabulator had performed well enough to be recognized in the mainstream, analog machines (with continuous motion) dominated the calculations during the war. If there was any aspect of art or aesthetics involved in the design of earlier mechanisms, it was crushed out by the war and replaced by the aspect of utility. Deadly accuracy was the need of the time. Similarly, inner effort was almost entirely devoted to formulating strategies. It was in the midst of this upheaval that a small circle was pulled together that would have a major role in the rest of the century: Veblen's circle.



As described in George Dyson's book, *Turing's Cathedral*, this small workforce of mathematicians employed by Oswald Veblen for calculating gun trajectories consisted of people who went on to make significant contributions to the development of computers.

> Veblen organized the teams of human computers who were placed under his command, introducing mimeographed computing sheets that formalized the execution of step-by-step algorithms for processing the results of the firing range tests. It took the entire month of February to fire the first forty shots, yet by May his group was firing forty shots each day, and the growing force of human computers was keeping up. Veblen recruited widely, with a knack for discovering future mathematicians and making the best use of their talents during the war. (*Turing's Cathedral*, p.40, 2012)

In addition, Veblen later helped set up the Institute for Advanced Study at Princeton (1930), which became the mecca for many talented mathematicians and physicists: Albert Einstein, John von Neumann, Kurt Gödel, Alan Turing, Claude Shannon, and Robert Oppenheimer. It was mainly in this Institute (and nearby Princeton University where Alan Turing worked) that the achievements of computing theory and technology were concentrated. The IAS was to computers what the School of Athens was for logic. It is quite illuminating to study this development in some detail, with attention to the concepts introduced by some of these workers.

Even though the mathematics for digital circuits was available since the late 1880s with the works of Boole, Frege, Peirce and others, it was only in 1937 that they were actually applied. Claude Shannon, who was aware of the works of Boole, realized that the algebra lent itself for use in switching circuits for telephone networks. Thus, a practical application of symbolic logic was determined for the first time. The effect of this discovery has been so influential that Shannon's thesis has been called "possibly the most important, and also the most famous, master's thesis of the century." Until Shannon, computing was carried out only with analog machines, e.g., the Differential Analyzer which was used to integrate differential equations necessary for missile and bomb trajectory determination. Only after the application to switching and logical circuitry was it possible to build a practical digital computer, marking the birth of the digital era.

This transition from analog to digital has a parallel in mathematics as well. As pointed out previously, the transition from Logic in general to Boolean Logic involved the sacrifice of infinite quantities, which cannot be represented either mathematically in an equation or mechanically. Even though all mathematical operations could be encoded, the numbers themselves had physical restrictions. This meant that all signatures of the concept of infinity had to disappear from the number line. Consider the number line below:

What are the categories of numbers possible? If a, b, c represent the digits, they can be classified as:

–Integers (a, b, c, -a, -b, -c etc.)
–Rational numbers (of the form a/b)
–Irrational numbers (5√a, 3√b etc. with a and b prime)
–Transcendental numbers (π, *e*, πa, *e*b, etc.)

Of these, only the first two can be measured exactly, and also constructed by ruler and compass. Most of the irrationals and all of the transcendentals cannot be calculated. Even though all these numbers exist on a number line one draws with the sweep of a pencil, only the first two sets allow manipulations and calculations. For example, there is no way to indicate "7√7" (7th root of 7) or "π" exactly on a number line, even though one can easily draw a continuous line from 1 to 2 or 3 to 4. This is the key principle that defines whether or not a number can be utilized in a mechanical process.

Machines can therefore be used to conduct operations on integers and rational numbers only, while for the other two, approximations are naturally involved. One can add, subtract, multiply and divide only using the integers and rational numbers, and the mechanisms which generate these functions together constitute an *analog computer*. Even ruler and compass, or folding in origami, is something that shows an analog operation.

Since irrationals, trigonometric functions, and logarithms are not constructible in general, for centuries the only possible way to obtain them has been through tabulating them empirically and looking them up when required. This meant that mathematicians spent considerable time in creating these tables for later use. These are not accessible to any known machine and are beyond the domain of *exact* calculation. Hence, the limits of analog computers are reached with rational numbers.

What are the numbers that can be represented digitally? As the name suggests, only those using the digits (fingers), namely the countable numbers. Negative numbers and decimal fractions cannot be represented directly in terms of 1s and 0s, they can only be encoded so that the sign of the number or the placement of the decimal is added as an additional string of digits. Nor can recurring decimals be represented, e.g., 1/3 can only be approximated as 0.3333333. All irrational numbers are impossible to calculate in finite time; this ultimately restricts the set of numbers to whole numbers, i.e., 0, 1, 2, … n, where “n” is as large a number as is physically possible to represent. This forms the range of a digital system, as a direct consequence of Boolean algebra. A digital computer can be 100% accurate only with integer addition, subtraction, and multiplication. Any other operation involves approximations.

This means that the range is reduced when one transitions from the ideal analog computer to the ideal digital computer. This mirrors the development of Boolean logic, which jettisons most of the comparisons except equality, and thus reduces the range of the copulas previously in use. Logic and Mechanism hence move in step with each other. The fields of mathematics generate the corresponding mechanical operations thus:

Geometry =>
Construction with compass/ruler =>
Analog machines

Arithmetic =>
Integer number operations =>
Digital machines

It was important to go into this much detail regarding this transition because, in addition to the discovery of the use of Boolean algebra in circuitry, several mathematical and logical identifications were made at this time with regard to computation. These were intimately connected to this transition into the digital world, since the focus now shifted from *geometric analog construction* to *arithmetical digital computation*. While the Greeks focused on finding out if a certain statement was *true*, developers of symbolic logic focused on finding out if a certain statement was *provable*, and formal mathematicians focused on finding out if a statement is *computable*. Concomitant with this, all the struggles of the Greeks with paradoxes in logic were recast as problems of proofs or problems in computability.

The early problems with logic can be shown with a classic example: the Liar’s paradox. This paradox (attributed, among others, to Epimenides) is generally stated as:

Epimenides (a Cretan) says: All Cretans are liars.

Is this a true or a false statement? If it is true, it implies that it is false, and if it is false, it becomes true. A stricter example is a sentence like this: *This sentence is false.* (Hehner 2014) Thus, right at the core of logic, a self-referential statement generates an absurdity, bringing into question the very ideas of true and false. This has been tackled in numerous ways for a long time by logicians. However, in the 20th century, two mathematicians recast the resolution of this problem into modern form and shifted focus from "truth" to "provability" and "computability." They were Kurt Gödel and Alan Turing, whose work appeared at the same time as Shannon's.

Kurt Gödel directed his attention to one of the questions posed by David Hilbert regarding formal systems of mathematics (1928). Hilbert asked whether such a formal description can ever have a complete and consistent set of rules. Since formal mathematics added several axioms, it was of primary importance for this system to be fully consistent if it was to be valid. Gödel found out that this was not possible and one could always find a statement that could not be proved within the system, making the system incomplete. This is the precise analogue of the Liar's Paradox cropping up once again because the Paradox was not resolved but shunted from one domain to the other.

The meaning of Gödel's Incompleteness Theorem (1931), as this came to be called, is clearly described in this warehouse analogy by David Black:

> To understand the consternation experienced by mathematicians as a result of Gödel's discovery of the essential incompleteness of mathematical systems, imagine that a mathematical system is a large warehouse, and that each true statement in the system is a box stored in the warehouse. Naturally, boxes are counted when they enter or leave the warehouse, and records are kept of the number of boxes and their locations in the warehouse. Now imagine taking a physical inventory of the contents of the warehouse; the point of such an inventory is to record the existence and location of every box contained by the warehouse. The people taking the inventory must devise a path through the warehouse so that they can count every box, missing none and counting none twice. Physical inventories involve no magic; they must simply be carefully planned and meticulously executed. Imagine that the inventory is now complete, and the records on the warehouse's contents are now available for inspection.
>
> Gödel's proof demonstrates a method whereby any wise-guy warehouse man, upon looking over the results of the physical inventory, can enter the warehouse and turn up a box that does not appear on the records. Suppose the inventory records are corrected to include the newly discovered box; after looking over the corrected records, the wise-guy can produce another unrecorded box, and can keep doing so without limit. If you were the manager of the warehouse, how would this make you feel about your operation? Gödel's incompleteness proof engendered similar feelings in mathematicians about their knowledge of their own mathematical systems. (David B. Black, *Context and Significance of Gödel's Proof*, Shoreline Vol. 4, p.55, 1991)

This theorem can be applied to describe the behavior of points on the number line exactly. Just as between any two numbers one can always find another number that wasn't counted before, in the set of algorithms, one can find one which wasn't recognized before.

While Gödel concentrated on whether a system of formal mathematics was complete and consistent, Alan Turing tried to find out if it was possible to say beforehand whether or not a certain statement could be proved (1937). Since a proof was intricately tied to mathematical mechanism, as shown earlier, generating a proof depended on whether a number was computable or not. And what did he mean by "computable"?

"*According to my definition, a number is computable if its decimal can be written down by a machine.*" For example, a recurring decimal is called computable because the rule to calculate it can be encoded into the machine. Similarly π and *e* can be expanded in terms of series expansions, and are called computable by Turing.

$$\pi = 4 (1 - 1/3 + 1/5 - 1/7 \ldots \infty)$$

It is at this juncture that the classic "bait-and-switch" can be identified: The computable number π is something that can be computed, as long as *infinite* time and memory are provided! This is really as meaningless as saying that it is possible to do something, with the minor problem that it would literally take forever to do. The very definition of computable numbers is fundamentally flawed, as they are by definition not computable. One can find a pattern to them, just as π generates a circular pattern, but that is a far cry from actually constructing a length π units long, which is not possible by calculation. Turing's concept of computable numbers does not address this issue (numbers that go to an infinite number of decimal places) but instead pushes it under the rug.

Secondly, the entire exercise of provability, paradoxes, computability, etc., can be illustrated by a simple example. Consider a group of people, say a dozen, and assume their talk is being studied by one interested researcher. There would be several statements which might be true, several lies, and several mixtures in their speech. If one among them declares aloud, "I am lying right now," he or she has uttered the Liar's Paradox. If the sentence is true, it is false (a lie), and if it is false, it is true, making it stuck in a loop. In a second scenario, assume that all the people are saying the exact same sentence. There is no way of saying if all of them are lying or if all of them are telling the truth.

These two situations exist, and the only way the truth of the matter can be adjudged is if there is a real event corresponding to what is being told by all of them and also experienced by the researcher. Since this depends on the experience of the researcher, logic necessarily has its limitations. This is the limit all logicians come up against.

However, Gödel converted the question of truth into one of provability and restated the Liar's Paradox in the language of formal logic with his Incompleteness Theorem. Turing's notion of computability was flawed, but nevertheless his attempt to reduce a system into computable and non-computable faced the same barrier, leading to his descriptions of the Halting Problem, which is "just the Liar's Paradox in fancy clothing." (Hehner, 2014) In addition, he also hit upon the same problem of verifying the truth of a statement, which cannot be done by staying within the logical system, be it by computing or by anything else. He therefore concluded that there was no way one could say beforehand by any method if a certain statement was computable or not.

Hence, this entire development had the effect of hiding the irrationals and transcendentals under the guise of computability, while the actual problems with logic were not resolved, but simply recast in a new language. The reason for the persistence of these logical problems was that the logic developed by the Greeks was still being used, only with more axioms and a smaller set of interrelationships (symbolic logic). The child inherited the defects of the parent, so to speak.

With the transition into digital machines, parallels with Greek development of logic are now mostly complete. They can be represented like this:

GREEK	MODERN
Aristotle	Boole
Stoics	Frege, Peirce
Epimenides	Gödel, Turing
Geometric construction	Digital computation
Euclidean geometry	Boolean algebra
Logic	Symbolic logic

It appears that this connection to Greek thought was missed by most people involved in the development of computers, who were really stimulated by Turing's work to build a machine that could compute anything. With the success of Shannon's circuits and Boolean logic, it was increasingly taken for granted that all thought processes can be expressed using Frege's symbolic logic, and that it can all be mechanized, even if it takes infinite time to arrive at the result (an illogical idea by all standards). Hence, the effects of irrationals and transcendentals were neglected and brushed to the side, while the essential problems that plagued Logic itself were not tackled directly, only reformulated.

Following the contributions of Shannon, Gödel, and Turing, the task of making digital machines feasible was taken up very effectively by John Von Neumann and his collaborators. This last leg of historical development, which continued up to the end of the century, will be described in the next chapter (to appear in *Research Bulletin* 22-2, Autumn/Winter 2017).

Editor's note: The references for this article, published in three segments, will appear at the end of the third installment (*Research Bulletin* 22-2 Autumn/Winter 2017).

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